NORTH SYDNEY GIRLS HIGH SCHOOL

2017 HSC Assessment Task 1

Mathematics Extension 2

General Instructions
- Reading Time – 2 minutes
- Working Time – 55 minutes
- Write using black pen.
- Board-approved calculators may be used
- In Questions 7−9 show relevant mathematical reasoning and/or calculations

Total Marks – 43

Section I Multiple Choice

5 marks
- Attempt Questions 1−5
- Allow about 8 minutes for this section

Section II Constrained Answer

10 marks
- Attempt Question 6
- Allow about 12 minutes for this section

Section III Free Response

28 marks
- Attempt Questions 7−9
- Allow about 35 minutes for this section

Student Name:________________________________

Student Number:______________________________

Teacher:

○ Ms Viswanathan ○ Ms Everingham
○ Ms Narayanan

QUESTION	MARK
1−5	/5
6	/10
7	/10
8	/9
9	/9
TOTAL	/43

Section I

5 marks
Attempt Questions 1–5
Allow about 8 minutes for this section

Use the multiple-choice answer sheet for Questions 1–5.

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1 What is $i-1$ in modulus-argument form?

(A) $\sqrt{2}\text{cis}\left(-\frac{\pi}{4}\right)$

(B) $\sqrt{2}\text{cis}\left(\frac{3\pi}{4}\right)$

(C) $2\text{cis}\left(-\frac{\pi}{4}\right)$

(D) $2\text{cis}\left(\frac{3\pi}{4}\right)$

2 Which of the following is equal to $\frac{3+2i}{3-2i}$?

(A) $\frac{5+12i}{13}$

(B) $\frac{13+12i}{13}$

(C) $\frac{5+12i}{5}$

(D) $\frac{13+12i}{5}$

3 What are the complex cube roots of -1?

(A) $-1,\ \frac{1}{2}\left(-1\pm i\sqrt{3}\right)$

(B) $1,\ \frac{1}{2}\left(1\pm i\sqrt{3}\right)$

(C) $-1,\ \frac{1}{2}\left(1\pm i\sqrt{3}\right)$

(D) $-1, -1, -1$

4 Given two complex numbers z_1 and z_2, $\arg z_1 = \theta_1$ and $\arg z_2 = \theta_2$, also $|z_1| = r_1$ and $|z_2| = r_2$.

When $|z_1 - z_2|$ has a maximum value $r_1 + r_2$, what is the relationship between θ_1 and θ_2?

(A) $|\theta_1 + \theta_2| = \pi$

(B) $|\theta_1 - \theta_2| = \pi$

(C) $|\theta_1 + \theta_2| = 2\pi$

(D) $|\theta_1 - \theta_2| = 2\pi$

5 Two complex numbers z_1 and z_2 with $z_2 \neq 0$ and $z_1 \neq z_2$, satisfy $\left|\dfrac{z_1 + z_2}{z_1 - z_2}\right| = 1$.

When considering $\dfrac{z_1}{z_2}$ which of the following statements is true?

(A) $\dfrac{z_1}{z_2}$ is purely real.

(B) $\dfrac{z_1}{z_2}$ is purely imaginary.

(C) $\dfrac{z_1}{z_2}$ is real and positive.

(D) None of the above need always be true.

End of Section I

Section II

10 marks
Attempt Question 6
Allow about 12 minutes for this section

Indicate your answer by entering it into the appropriate diagrams on the answer sheet provided

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Question 6 (10 marks) Use the answer sheet provided.

(a) The complex number z is such that $|z| = 2$ and $\arg(z) = \dfrac{\pi}{4}$.

(i) Plot and label the point *P* which represents z on the Argand diagram provided.

On the same diagram also plot and label the points:

(ii) *Q* representing z^2 **1**

(iii) *R* representing $\dfrac{1}{2}iz$ **1**

(iv) *S* representing $\dfrac{1}{2}iz - \overline{z}$ **1**

(b) Sketch the following loci on separate diagrams on the answer sheet provided.

(i) $\dfrac{\pi}{3} < \arg z \le \dfrac{2\pi}{3}$ and $\operatorname{Im}(z) - \sqrt{3}\operatorname{Re}(z) \le 2$ **2**

(ii) $|z| < |z - 2 + i|$ **2**

(iii) (α) $|z - 2i| \le 1$ **2**

(β) Given that z is a complex number satisfying $|z - 2i| \le 1$, write down the least positive argument of z. **1**

End of Section II

Section III

28 marks
Attempt Questions 7–9
Allow about 35 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing papers are available.

In Questions 7 - 9, your responses should include relevant mathematical reasoning and/or calculations.

Question 7 (10 marks) Use a SEPARATE writing booklet.

(a) If $z = 5 - 2i$ and $w = -4 + i$

(i) Find $\text{Im}(4iz - 3)$. **1**

(ii) Find $\overline{w} + 2z$. Express your answer in the form $a + ib$, where a and b are real numbers. **1**

(b) The equation $2x^2 - kx + 17 = 0$ has one complex root, α, such that $\text{Re}(\alpha) = \frac{5}{2}$. If k is real, find both roots of the equation and the value of k. **3**

(c) Let $z = 1 - i$ and $w = -1 + i\sqrt{3}$. The argument of z is $\left(-\frac{\pi}{4}\right)$.

(i) Find the argument of w. **1**

(ii) Hence find wz in modulus- argument form. **2**

(iii) Hence prove that $\sin\frac{5\pi}{12} = \frac{\sqrt{2}\left(1+\sqrt{3}\right)}{4}$. **2**

Question 8 (9 marks) Use a SEPARATE writing booklet.

(a) (i) Find the complex square roots of $-24+10i$, in the form $a+ib$. **3**

(ii) Hence, or otherwise, solve $z^2-(3+i)z+8-i=0$. **2**
Give your answers in the form $a+ib$.

(b) Let $\omega_1, \omega_2, \omega_3, \omega_4$ and ω_5 be the solutions of $z^5=1$. Suppose that z is a complex number such that $|z|=1$.

(i) Explain why $\omega_1+\omega_2+\omega_3+\omega_4+\omega_5=0$. **1**

(ii) Show that $|z-\omega_i|^2=(z-\omega_i)(\overline{z}-\overline{\omega_i})$ where $i=1,2...5$. **1**

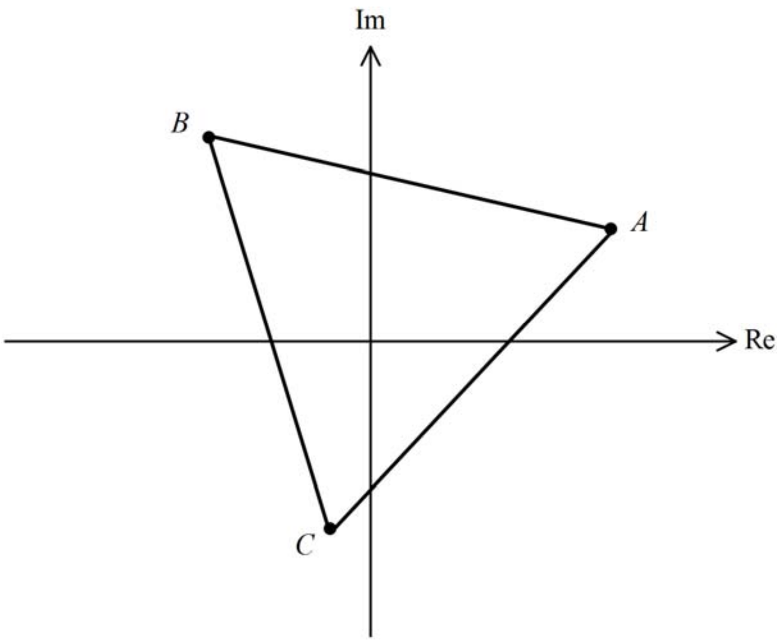
(iii) By using the results in parts (i) and (ii), prove that $\sum_{i=1}^{5}|z-\omega_i|^2=10$. **2**

Question 9 (9 marks) Use a SEPARATE writing booklet.

(a) (i) If $z = \cos\theta + i\sin\theta$, show that $z^n + z^{-n} = 2\cos n\theta$. **2**

(ii) Hence, deduce that $\cos\theta\cos 2\theta = \frac{1}{2}(\cos 3\theta + \cos\theta)$. **3**

(b) On the Argand diagram, the points *A*, *B* and *C* represent the complex numbers u, v and w respectively. ΔABC is equilateral, named with its vertices taken anticlockwise.



(i) Show that $w - u = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)(v - u)$. **2**

(ii) Show that $u^2 + v^2 + w^2 = uv + vw + wu$. **2**

End of Paper

Name:______________________________ Mathematics Class: __________

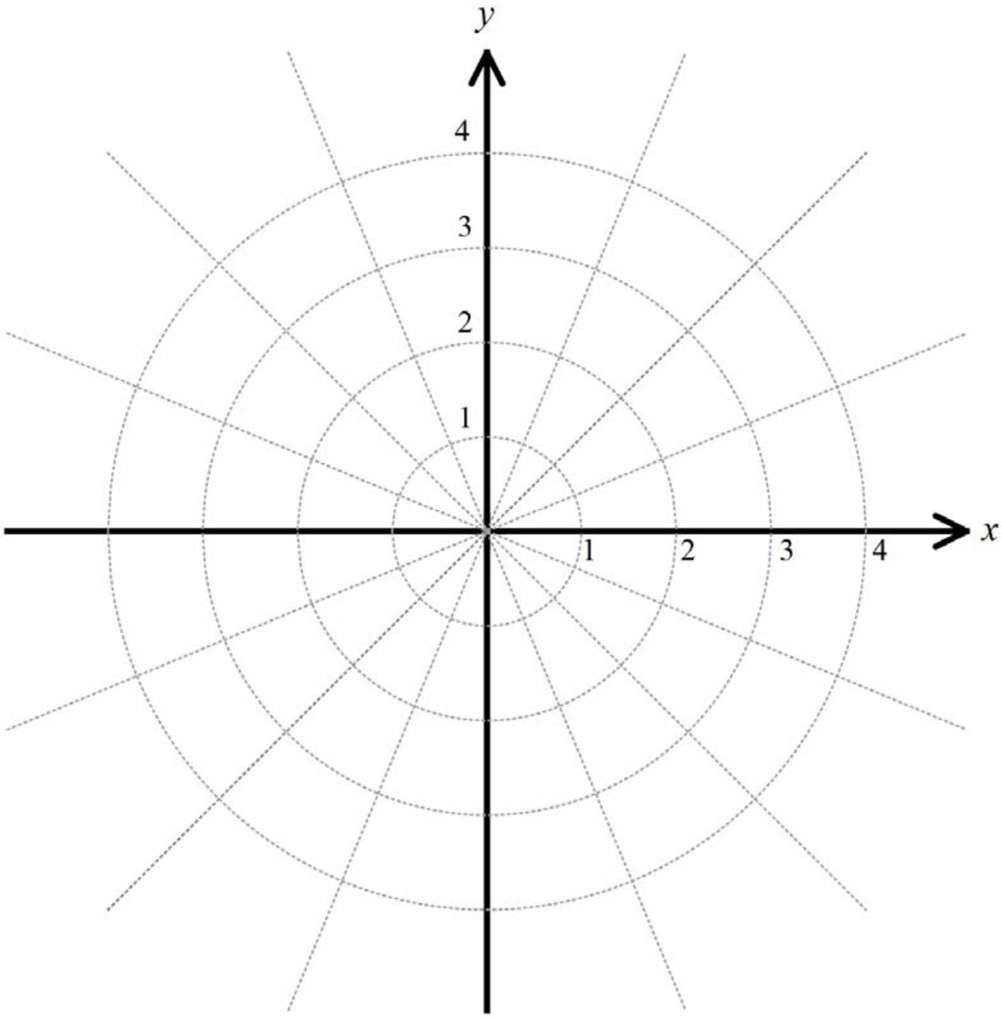
Constrained Answer Sheet – Write your answer in the space provided.

6. (a) (i) P represents z

(ii) Q represents z^2

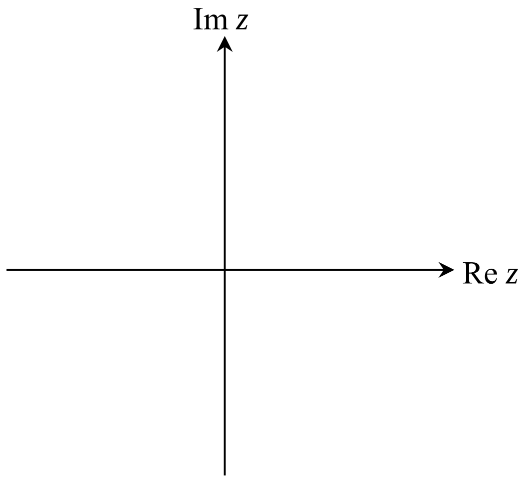
(iii) R represents $\frac{1}{2}iz$

(iv) S represents $\frac{1}{2}iz - \overline{z}$

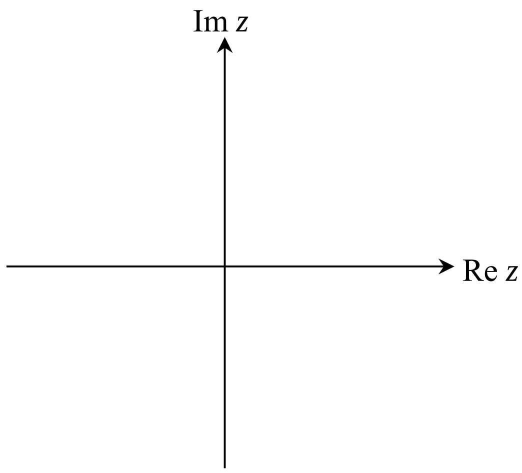


Name:_______________________________ Mathematics Class: __________

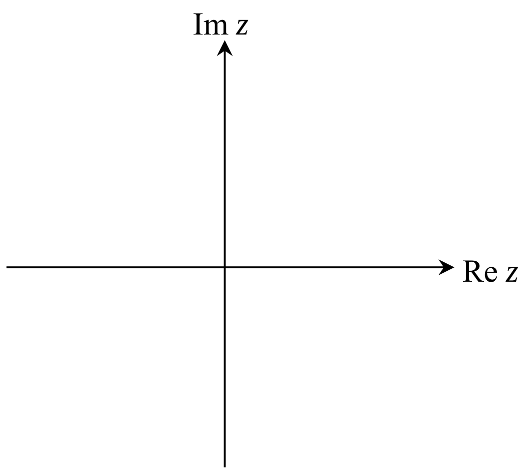
(b) (i) $\frac{\pi}{3} < \arg z \le \frac{2\pi}{3}$ and $\mathrm{Im}(z) - \sqrt{3}\,\mathrm{Re}(z) \le 2$



(ii) $|z| < |z - 2 + i|$



(iii) (α) $|z - 2i| \le 1$



(β) Given that z is a complex number satisfying $|z - 2i| \le 1$, write down the least positive argument of z.

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2017 HSC Task 1 Mathematics Extension 2.

SECTION 1.

1. B $\quad \text{mod}(-1+i) = \sqrt{2}$, $\arg(1-i) = \frac{3\pi}{4}$

2. A $\quad \frac{(3+2i)(3+2i)}{13} = \frac{5+12i}{13}$

3. C

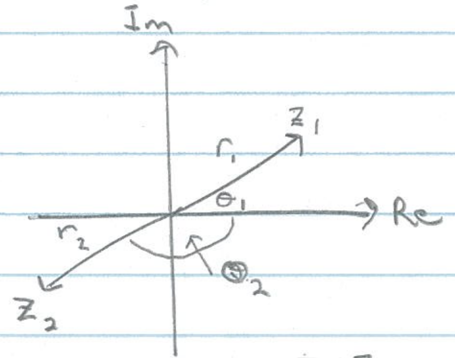
$z^3 = -1$

$z^3 = \text{cis}(\pi + 2k\pi)$

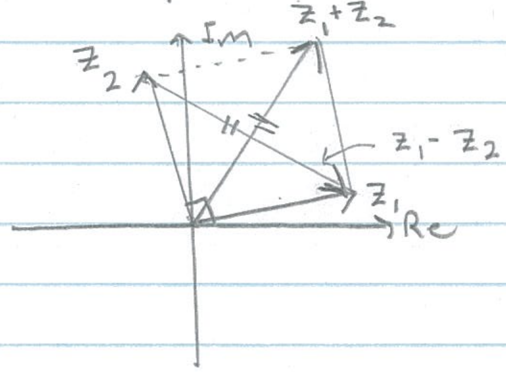
$z = \text{cis}\frac{\pi}{3}, \text{cis}\left(-\frac{\pi}{3}\right), \text{cis}(\pi)$

$z = \frac{1}{2} + \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i, -1$

4. B



5. B



diagonals of parallelogram are equal

∴ rectangle.

$z_2 = kiz_1$

$\frac{z_1}{z_2} = \frac{1}{ki} = \frac{-i}{k}$

purely imaginary.

Name:________________________________ Mathematics Class: __________

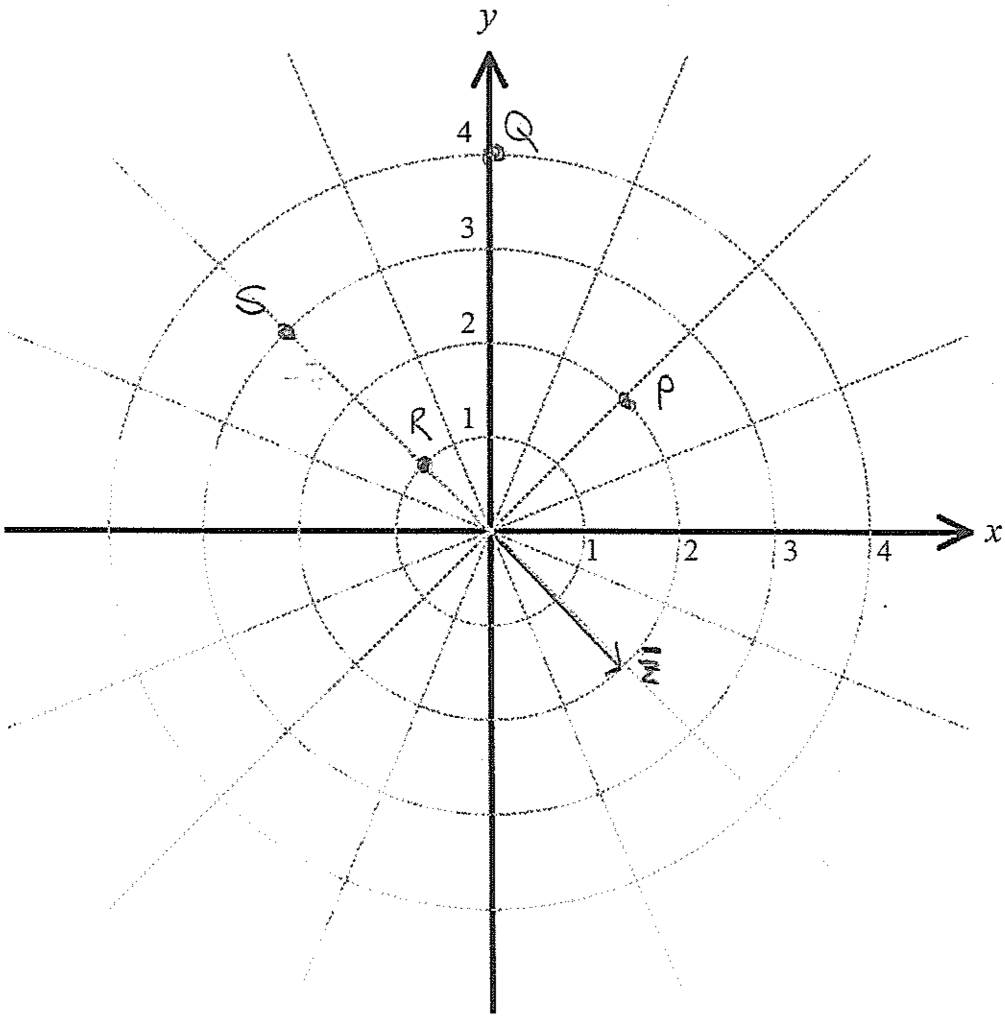
Constrained Answer Sheet – Write your answer in the space provided.

6. (a) (i) P represents z $2\text{cis}\left(\frac{\pi}{4}\right)$

(ii) Q represents z^2 $4\text{cis}\left(\frac{\pi}{2}\right)$

(iii) R represents $\frac{1}{2}iz$ $\frac{1}{2}\text{cis}\left(\frac{\pi}{2}\right) \times 2\text{cis}\left(\frac{\pi}{4}\right) = \text{cis}\left(\frac{3\pi}{4}\right)$

(iv) S represents $\frac{1}{2}iz - \bar{z}$ $\frac{1}{2}\text{cis}\left(\frac{\pi}{2}\right) - 2\text{cis}\left(-\frac{\pi}{4}\right)$



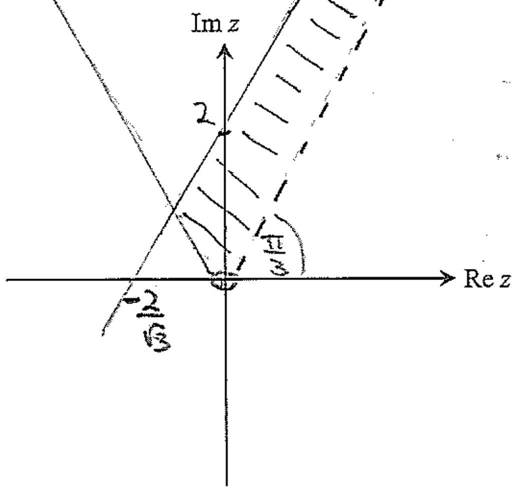
Please turn over….

Name:______________________________ Mathematics Class: __________

(b) (i) $\frac{\pi}{3} < \arg z \leq \frac{2\pi}{3}$ and $\text{Im}(z) - \sqrt{3}\,\text{Re}(z) \leq 2$

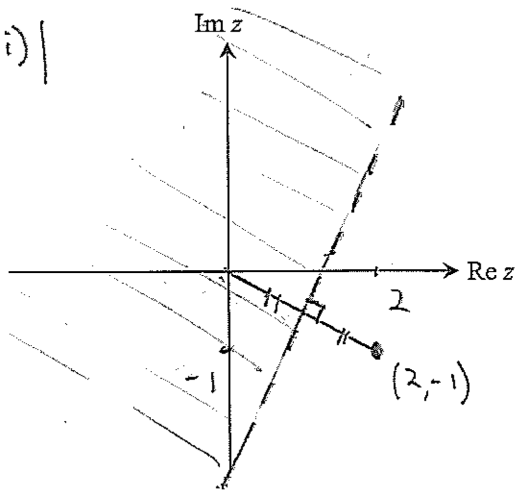
$y - \sqrt{3}x \leq 2$

$y \leq \sqrt{3}x + 2.$

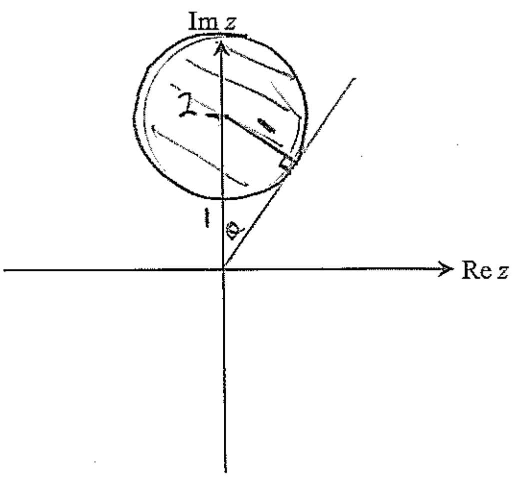


(ii) $|z| < |z - 2 + i|$

$|z| < |z - (2 - i)|$



(iii) (α) $|z - 2i| \leq 1$



$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$

(β) Given that z is a complex number satisfying $|z - 2i| \leq 1$, write down the least positive argument of z.

$\arg z = \frac{\pi}{3}$

Question 7

a) (i) $z = 5 - 2i$ $\quad w = -4 + i$

$4iz - 3 = 4i(5 - 2i) - 3$
$= 20i + 5$

$\text{Im}(4iz - 3) = 20.$

(ii) $\bar{w} + 2z = -4 - i + 2(5 - 2i)$
$= 6 - 5i$

a) Generally well done.

b) $2x^2 - kx + 17 = 0$

Let the roots be $\begin{cases} \frac{5}{2} + iy \\ \frac{5}{2} - iy \end{cases}$ (real coefficients ∴ conjugate roots)

product of roots: $\frac{25}{4} + y^2 = \frac{17}{2}$

$y^2 = \frac{9}{4}$

$y = \frac{3}{2}.$

∴ roots are $\frac{5}{2} + \frac{3}{2}i$, $\frac{5}{2} - \frac{3}{2}i$

sum of roots. $\frac{10}{2} = \frac{k}{2}$

∴ $k = 10.$

b) Some students failed to note that *k* was given to be real. Consequently they assumed two roots α and β. This made the subsequent working more complicated and sadly fruitless. A maximum of 1 mark was awarded to such responses. Common errors included not accounting for the non-monic when making relationships of sum and product of roots.

Question 7 cont.

c) $z = 1 - i \qquad w = -1 + i\sqrt{3}$

i) $\tan(\arg w) = -\sqrt{3}$

$\arg w = \frac{2\pi}{3}$

(ii) $\arg wz = \arg w + \arg z$

$= \frac{2\pi}{3} - \frac{\pi}{4}$

$= \frac{5\pi}{12}$

(iii) $wz = (-1 + i\sqrt{3})(1 - i)$

$= -1 + i\sqrt{3} + i + \sqrt{3}$

$= (-1 + \sqrt{3}) + i(1 + \sqrt{3})$ ①

$wz = 2\text{cis}\left(\frac{2\pi}{3}\right) \cdot \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$

$= 2\sqrt{2}\,\text{cis}\left(\frac{5\pi}{12}\right)$ ②

equating imaginary parts of ① & ②

$1 + \sqrt{3} = 2\sqrt{2}\sin\left(\frac{5\pi}{12}\right)$

$\sin\left(\frac{5\pi}{12}\right) = \frac{\sqrt{2}(1 + \sqrt{3})}{4}$

c) Generally well done. Some students failed to match the exact expression in the question for instance denominator not rationalised or numerator not factorised. In a “show that” or “prove” question your response is not complete until the exact expression is obtained.

Question 8

a) (i) Let $z^2 = -24 + 10i$

$(x+iy)^2 = -24 + 10i$

$x^2 - y^2 + 2xyi = -24 + 10i$

Equating real & imaginary parts.

$x^2 - y^2 = -24$ ①

$2xy = 10$

$xy = 5$ ②

By inspection $x = 1$, $y = 5$.

$x = -1$, $y = -5$.

$\therefore$ the 2 square roots are $1+5i$ & $-1-5i$

(ii) $z^2 - (3+i)z + (8-i) = 0$.

$\Delta = (3+i)^2 - 4(8-i)$

$= 9 - 1 + 6i - 32 + 4i$

$= -24 + 10i$

From (i) square roots of $(-24+10i)$ are $\pm(1+5i)$

$\therefore z = \dfrac{(3+i) \pm (1+5i)}{2}$

$= \dfrac{4+6i}{2}, \dfrac{2-4i}{2} \Rightarrow 2+3i, 1-2i$

Question 8 cont.

b) i) $\omega_1, \omega_2 \ldots \omega_5$ are the roots of $z^5 - 1 = 0$

By sum of roots $\omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 = 0$

(coefficient of z^4 is 0)

(ii) $|z - \omega_i|^2 = (z - \omega_i)(\overline{z - \omega_i})$

since $|z|^2 = z . \bar{z}$

$= (z - \omega_i)(\bar{z} - \overline{\omega_i})$ for $i = 1, 2 .. 5$

(iii) $\sum_{i=5}^{5} |z - \omega_i|^2 = \sum_{i=1}^{5} (z - \omega_i)(\bar{z} - \bar{\omega}_i)$

from part (ii)

$= \sum_{i=1}^{5} (z\bar{z} - \omega_i \bar{z} - z\overline{\omega_i} + \omega_i \overline{\omega_i})$

$= \sum_{i=1}^{5} (1 - z\overline{\omega_i} - \omega_i \bar{z} + 1)$

since $z\bar{z} = |z|^2 = 1$
$\omega_i \overline{\omega_i} = |\omega_i|^2 = 1$

$= \sum_{i=1}^{5} (2 - z\overline{\omega_i} - \omega_i \bar{z})$

$= 10 - z \sum_{i=1}^{5} \overline{\omega_i} - \bar{z} \sum_{i=1}^{5} \omega_i$

$= 10 - \bar{z}(0) - \bar{z}(0)$

since $\omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 = 0$

$= 10$

$\overline{\omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5} = \bar{0}$
$\overline{\omega_1} + \overline{\omega_2} + \overline{\omega_3} + \overline{\omega_4} + \overline{\omega_5} = 0$

Question 9

a) $z = \cos\theta + i\sin\theta$

(i) $z^n + z^{-n} = (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n}$

$= \cos(n\theta) + i\sin(n\theta) + \cos(-n\theta) + i\sin(-n\theta)$

By De Moivre's Theorem

$= \cos n\theta + i\sin(n\theta) + \cos(n\theta) - i\sin(n\theta)$

since $\cos x = \cos(-x)$

$\sin(-x) = -\sin x.$

$= 2\cos n\theta.$

(ii) From (i) $z^1 + z^{-1} = 2\cos\theta$, $z^2 + z^{-2} = 2\cos 2\theta$

$\cos\theta = \dfrac{z^1 + z^{-1}}{2}$ $\cos 2\theta = \dfrac{z^2 + z^{-2}}{2}.$

$\therefore$ LHS $= \cos\theta\cos 2\theta$

$= \dfrac{z^1 + z^{-1}}{2} \times \dfrac{z^2 + z^{-2}}{2}$

$= \dfrac{z^3 + z + z^{-1} + z^{-3}}{4}$

$= \frac{1}{4}\left(z^3 + z^{-3} + z^1 + z^{-1}\right)$

$= \frac{1}{4}\left(2\cos 3\theta + 2\cos\theta\right)$ from (i)

$= \frac{1}{2}\left(\cos 3\theta + \cos\theta\right)$

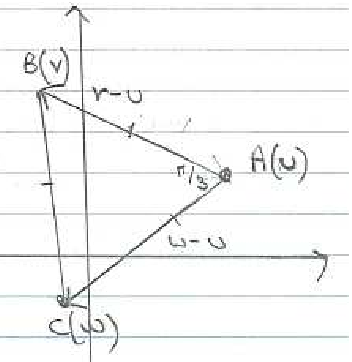
a) (i) This should have been standard bookwork. Many responses were sloppy with inadequate explanation of how one line of working is derived from the previous one. However, the marking was generous with minor penalties for missing steps. Lack of reasoning was largely not penalised.

(ii) Several different approaches were successful even if not always efficient. The most efficient approach was to use the result in (i) to make a suitable substitution, expand and then use the result again as shown in the solutions. Many students used the double angle result for $\cos 2\theta$ in the LHS and then manipulated the RHS to match the same expression. The compound angle result for $\cos 3\theta$ is not a quotable result.

Question 9 cont.

b)

(i) $\overrightarrow{AB}$ represents $v-u$,
$\overrightarrow{AC}$ represents $w-u$.



Now,
$\angle BAC = \frac{\pi}{3}$ (ΔABC is equilateral)
also $|AB| = |AC|$

$\text{cis}\frac{\pi}{3} \times (v-u)$

means multiply modulus of $(v-u)$ by 1 & rotate the argument by $\frac{\pi}{3}$ anticlockwise.

This yields $\overrightarrow{AC}$ which is $w-u$.

$\therefore\ w-u = \text{cis}\left(\frac{\pi}{3}\right)(v-u)$

b) (i) Most students were able to attempt some explanation of this result. However, most explanations were missing some element. Commonly, students failed to explain that the two vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ have the same lengths or to state that $\overrightarrow{AC}$ is an anti-clockwise rotation of $\overrightarrow{AB}$ by $\frac{\pi}{3}$.

(ii) $w-u = \text{cis}\left(\frac{\pi}{3}\right)(v-u) \Rightarrow \frac{w-u}{v-u} = \text{cis}\frac{\pi}{3}$ (1)

similarly,
$v-w = \text{cis}\left(\frac{\pi}{3}\right)(u-w) \Rightarrow \frac{v-w}{u-w} = \text{cis}\frac{\pi}{3}$ (2)

since $\overrightarrow{CB}$ is an anticlockwise rotation of $\overrightarrow{CA}$ by $\frac{\pi}{3}$.

Equating (1) & (2)

$\frac{w-u}{v-u} = \frac{v-w}{u-w} \Rightarrow (w-u)(u-w) = (v-u)(v-w)$

$2wu - u^2 - w^2 = v^2 - uv - wv + uw$

$\therefore\ u^2 + w^2 + v^2 = uv + wv + uw$

b)(ii) Alternate solution

$|AB| = |BC| = |AC|$ since ΔABC is equilateral

$\therefore |AC|^2 = |AB|.|BC|$ since $|z_1 z_2| = |z_1||z_2|$

$|(w-u)(w-u)| = |(v-u)(w-v)|$

$|w^2 + u^2 - 2uw| = |vw - v^2 - uw + uv|$

$w^2 + u^2 - 2uw = vw - v^2 - uw + uv$ is a solution.

$u^2 + v^2 + w^2 = uv + vw + uw$ as required.

(ii) Poorly done. Many students attempted to use the cosine rule not realising that the cosine rule applies to lengths not vectors. Others tried to use the result $(a+b+c)^2 = a^2+b^2+c^2+2ab+2ac+2bc$. In general, it was not possible to show the required result without using part (i) and extending the result in (i) to a second pair of sides.